Paweł Nurowski

Differential Geometry

Fall 2008/2009

Stony Brook University
Math Department

L.1

S.B. 2.09.2008

Notation

① M - n dimensional **manifold** (smooth)

$p, q \in M$, $p \neq q$ $\Rightarrow \exists\, U_a, U_b$ s.t. $U_a \cap U_b = \phi$ and $p \in U_a$, $q \in U_b$

M is a topological Hausdorff, paracompact space

(e.g. if it has a countable basis for its topology then it is paracompact)

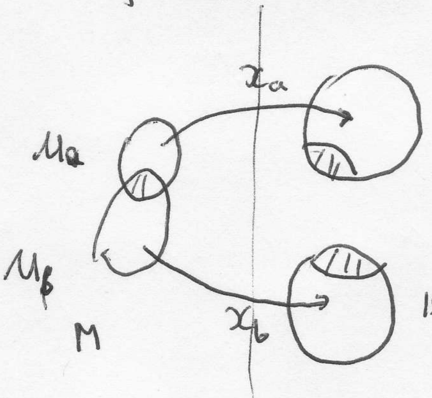
$M = \bigcup_{a \in I} U_a$ $\qquad U_a$ - open sets

(U_a, x_a) $\qquad x_a : U_a \longrightarrow \mathbb{R}^n$ homeomorphism $\qquad p \ni U_a \quad x_a(p) = (x^\mu)\ \ \mu = 1, \dots n$

chart

domain of a chart

local coordinates



$$x_b \circ x_a^{-1} \Big|_{x_a(U_a \cap b)}$$

$$x_a \circ x_b^{-1} \Big|_{x_b(U_a \cap U_b)}$$

are both smooth (have all partial derivatives)

$\mathcal{A} = \{ (U_a, x_a),\ a \in I \}$ - atlas

② **Differentiable map**

M, N - manifolds $\qquad \phi : M \to N$ is differentiable of class k

$\forall\ (U, x) \in \mathcal{A}(M)$ $\qquad y \circ \phi \circ x^{-1}$ is of class k
$\quad (V, y) \in \mathcal{A}(N)$

- $M \subset \mathbb{R}^1 \Rightarrow \phi$ is called a curve

- $N \subset \mathbb{R}^1 \Rightarrow \phi$ is called a function

③ Tangent vector

$p \in M$, $\mathcal{F}(p)$ - algebra of ~~of~~ functions of class C^∞ defined in an neighbourhood of p.

$$\mathcal{F}(p) = \{ f : U_p \to \mathbb{R}^1,\ f \text{ of class } C^\infty \}$$

Two curves $\gamma, \tilde{\gamma}$ of class C^1 are <u>tangent</u> at $p = \gamma(0) = \tilde{\gamma}(0)$

iff $\quad \forall f \in \mathcal{F}(p) \quad \frac{d}{dt} f \circ \gamma(t) \Big|_{t=0} = \frac{d}{dt} f \circ \tilde{\gamma}(t) \Big|_{t=0}$

This is an equivalence relation on the set of curves passing through p.

<u>Tangent vector</u> ~~to γ~~ at p is an equivalence ~~an equivalence~~ class of curves tangent at p ~~with γ~~.

This defines a map $\quad X : \mathcal{F}(p) \longrightarrow \mathbb{R}$

$$X(f) = \frac{d}{dt}\left[f \circ \gamma(t)\right]\Big|_{t=0}$$

- Local representation: $\quad x$ - local coord. around $p \quad x \circ \gamma(t) = (x^\mu(t))$

$$X(f) = \frac{d}{dt} f \circ \gamma(t) \Big|_{t=0} = \frac{d}{dt} \left(f \circ x^{-1}\right) \circ (x \circ \gamma)(t) \Big|_{t=0} =$$

$$= \frac{\partial f}{\partial x^\mu}\Big|_p \frac{dx^\mu}{dt}\Big|_{t=0} = X^\mu \frac{\partial f}{\partial x_\mu}\Big|_p$$

$$X^\mu = \frac{dx^\mu}{dt}\Big|_{t=0}.$$

$$X = X^\mu \frac{\partial}{\partial x^\mu}\Big|_p \qquad (X^\mu) \in \mathbb{R}^n$$

- <u>Properties</u>: $\quad X : \mathcal{F}(p) \longrightarrow \mathbb{R}$

  1° X linear

  2° $X(f \cdot g) = X(f) g(p) + f(p) X(g)$

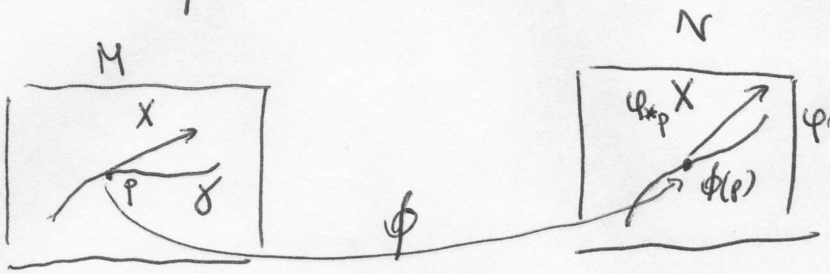
  Leibniz rule.

④ Tangent space $T_p(M)$ at p

vector space of all X as above. Locally $\left(\frac{\partial}{\partial x^\mu}\Big|_p\right)$ $\mu = 1, \ldots, n$, basis in $T_p(M)$.

⑤ Transport of tangent vectors / differential of a map.

$$\phi : M \longrightarrow N \qquad \text{differentiable map.}$$



Example 1 e.g. $\phi : (x,y,z) \longmapsto (y^2, x^3, z+x)$

$$X = A\partial_x + B\partial_y + C\partial_z$$

$$\gamma(t) = (At+x_0,\ Bt+y_0,\ Ct+z_0) \qquad \varphi(\gamma(t)) = \left((Bt+y_0)^2, (At+x_0)^3, (C+A)t+x_0+z_0\right)$$

$$\varphi_{*(x_0,y_0,z_0)} X = 2(Bt+y_0)B\big|_{t=0}\partial_x + 3(At+x_0)^2A\big|_{t=0}\partial_y + (C+A)\big|_{t=0}\partial_z =$$

$$= 2By_0\,\partial_x + 3Ax_0^2\,\partial_y + (C+A)\,\partial_z$$

$$\varphi_{*(x_0,y_0,z_0)} : \begin{pmatrix} A \\ B \\ C \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} 0 & 2y_0 & 0 \\ 3x_0^2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}}\begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

$$\varphi_{*(x_0,y_0,z_0)} = \frac{\partial \phi^i}{\partial x^\mu}\bigg|_{(x_0,y_0,z_0)}$$

Properties

- ϕ_{*p} is linear
- locally $\left(\frac{\partial}{\partial x^\mu}\Big|_p\right)$ basis in $T_p(M)$ $\quad \mu = 1, \dots m$

  $\left(\frac{\partial}{\partial y^i}\Big|_{\varphi(p)}\right)$ basis in $T_{\varphi(p)}(N)$ $\quad i = 1, \dots n$

$$\varphi_{*p}\frac{\partial}{\partial x^\mu}\bigg|_p = \underbrace{\frac{\partial \varphi^i}{\partial x^\mu}\bigg|_p}\frac{\partial}{\partial y^i}\bigg|_{\varphi(p)}$$

$$\varphi_{*p} = d\phi_p$$

$\varphi_{*p} \sim \frac{\partial \varphi^i}{\partial x^\mu}\Big|_p \Rightarrow$ hence the name differential.

⑥ Immersions and embeddings.

$\phi : M \to N$ is an immersion if ϕ_{*p} is injective for all $p \in M$.

Example 1 continued

ϕ is not an immersion in $\mathbb{R}^3$ since it is not injective

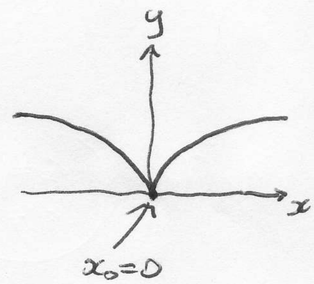
when either $x_0 = 0$ or $y_0 = 0$.

Ex 2

$\gamma : \mathbb{R} \to \mathbb{R}^2 \qquad x \longmapsto (x^3, x^2)$ has

$$\varphi_{* x_0} = (3x_0^2, 2x_0)$$

not an immersion.



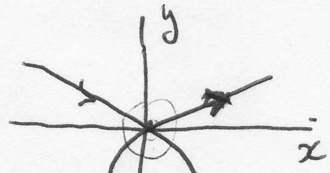
An immersion $\phi : M \to N$ is an embedding if ϕ is a homeomorphism onto $\phi(M) \subset N$

(topology on $\phi(M)$ induced from N)

Ex 3

$\gamma : \mathbb{R} \to \mathbb{R}^2 \qquad x \longmapsto (x^3 - 4x, x^2 - 4)$

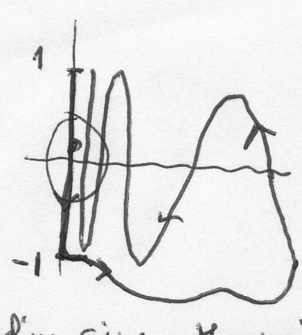
$$\varphi_{* x_0} = (3x_0^2 - 4, 2x_0)$$ is an immersion



Not embedding because of selfintersection

Ex 4

$$\gamma(t) = \begin{cases} (0, -(x+2)) & x \in (-3, -1) \\ \text{regular curve} & x \in (-1, -\frac{1}{\pi}) \\ (-x, -\sin\frac{1}{x}) & x \in (-\frac{1}{\pi}, 0) \end{cases}$$



immersion, but not embedding since the neighb. of a point on vertical line consists of disjoint interv.

(7) Submanifold.

If $M \subset N$ and the inclusion is an embedding then M is called submanifold of N.

(8) Codimension

If M is a submanifold of N and $\dim M = m$, $\dim N = n$ then $n-m$ is called a codimension of M in N.

Hypersurface a submanifold of codimension 1.